PHIL 4770: Philosophy of Mathematics
Aspirational Course Outline

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March 29, 2025

The twentieth century witnessed a dramatic transformation of the philosophy of mathematics. This transformation might be described, with some whimsy, as the march from the Continuum Hypothesis, the first of twenty-three problems on Hilbert's famous list, which he communicated in an address to the second International Congress of Mathematicians held in Paris in 1900, to the question, "$\mathsf{P} = \mathsf{NP}$?", among the Clay Mathematics Institute's seven Millennium Prize Problems, posted to the web in 2000. Cantor's Continuum Hypothesis spurred research in set theory through the end of the century, though its fate was rather different than any Hilbert might ever have anticipated. The question, "$\mathsf{P} = \mathsf{NP}$?", adumbrated in a 1956 letter from Gödel to von Neumann, was explicitly posed in the work of Cook, Karp, and Levin around 1970, and continues to be the focus of intense investigation by computer scientists and mathematicians. Both problems, and the research programs they spawned, have much to offer by way of material for philosophical reflection.

Though we will focus on twentieth (and twenty-first) century topics in the philosophy and foundations of mathematics, a guiding theme for our investigation will be a problem of ancient origin, Zeno's "Paradox of Measure": how can a line segment with extension be composed of extensionless points. We will stress points of contact between ancient views on the nature of the linear continuum and the late nineteenth-century "arithmetization of analysis" undertaken by mathematicians such as Richard Dedekind. In pursuing this theme, we will be led to see the relevance of some topics of current research interest, such as cardinal characteristics of the continuum and the reverse mathematics of weak second-order theories, to unravelling millennial puzzles surrounding infinite divisibility.

The following outline is very much *aspirational* – there is far too much listed than could possibly be covered in one term. Selection of topics will be informed by the interests of participants in the class.

1. The Arithmetization of Analysis (Reading: Richard Dedekind, *Continuity and Irrational Numbers*)
   (a) From antiquity: irrationality of $\sqrt{2}$
   (b) Dedekind Cuts: an order-complete, separable, dense linear order
   (c) Categoricity of the second-order theory of $\langle \mathbb{R}, < \rangle$
2. The Origins of Set Theory (Reading: John C. Oxtoby, *Measure and Category*)
   (a) The unit interval is uncountable
      i. Via order-completeness
      ii. Via outer measure
      iii. Via the Cantor Diagonal Theorem
   (b) The Cantor-Bendixson Theorem - the use of transfinite iteration in the analysis of closed sets of reals establishing that every closed uncountable set of reals has a non-empty perfect subset
   (c) Cantor's Continuum Hypothesis (CH): If $X \subseteq \mathbb{R}$ is uncountable, then $|X| = 2^{\aleph_0}$. (Hilbert's First Problem)
3. The Development of Set Theory (Reading: Thomas Jech, *Set Theory*)
   (a) The paradoxes
   (b) Zermelo's axiomatization of set theory (Z)
      i. The cumulative hierarchy up to $\omega + \omega$: $V_{\omega+\omega} \models$ Z
      ii. The well-ordering principle (AC)
      iii. The axiom of replacement: $V_\kappa \models$ ZFC, if κ is strongly inaccessible.
   (c) Gödel's Inner Model of Constructible Sets: the relative consistency of the the Axiom of Choice and the Generalized Continuum Hypothesis

   i. ZF + the axiom of constructibility (V=L) is consistent relative to ZF.
   ii. ZF + V=L entails AC and GCH.

(d) Large cardinal axioms that entail V $\neq$ L

(e) The formal independence of CH from ZFC: Cohen's method of forcing

(f) Cardinal Characteristics of the Continuum

4. Hilbert's Program (David Hilbert, *Address to the International Congress of Mathematicians, 1900*, *On the Infinite*, and *The Foundations of Mathematics*)

5. The Formalization of Logic (Reading: Jon Barwise, *Handbook of Mathematical Logic* Chapter A.1; David Marker, *Model Theory*, Chapters 1-2; Kees Doets, *Uniform short proofs for classical theorems*)

   (a) From Frege to first-order logic
   (b) The Gödel Completeness Theorem
   (c) The Compactness Theorem
   (d) The Löwenheim-Skolem Theorem
   (e) Lindstrom's Theorem

6. Decidable Theories (Reading: David Marker, *Model Theory*, Chapter 3)

   (a) Presburger Arithmetic
   (b) Real Closed Fields

7. The Incompleteness of Mathematical Theories and Undecidability Phenomena (Reading: Herbert Enderton, *A Mathematical Introduction to Logic*, Chapter 3, Sections 3.3-3.5; Craig Smorynski, *Handbook of Mathematical Logic* Chapter D.1; Alan Turing, *On computable numbers with an application to the* Entscheidungs *problem*; Kurt Gödel, *Letter to John von Neumann*)

   (a) Gödel's First Incompleteness Theorem

   (b) Gödel's Second Incompleteness Theorem (Hilbert's Second Problem)
   (c) Turing's explication of mechanical computability and the Church-Turing Thesis
   (d) The undecidability of the *Entscheidungs* Problem
   (e) Hilbert's Tenth Problem
   (f) Computational Complexity Theory

8. Reverse Mathematics (Reading: Stephen Simpson, *Subsystems of Second-Order Arithmetic*, Chapters 1,2, and 4; Denis Hirschfeldt, *Slicing the Truth*, Chapters 1-4)

Requirements

The final grade for the course will be based on completion of biweekly exercises. These will consist in problem sets and brief essay responses to prompts based on the readings and lectures. Students will be strongly encouraged to present solutions to exercises or lead discussions of exercises and prompts in class. Regular attendance and active class participation is mandatory.

Prerequisites

There is no formal prerequisite. Experience with mathematical logic and "proof-based" mathematics would be useful.